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Note

A solution for the coloured cubes problem

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Abstract

Let C be any set of q cubes in which every face in each one of them has to be coloured using one colour in a set K of q colours. It is asked how to raise, if it is possible, a pile with the q cubes in such a way that every colour will appear once in every one of the four faces of the pile. The case $q = 4$ was solved long time ago. Now, an answer is presented for the general case by means of an efficient algorithm. This method is based on a particular linear program which always produces integer solutions. © 1999 Elsevier Science B.V. All rights reserved.

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1. The coloured cubes problem

Let $C = \{c_1, c_2, \ldots, c_q\}$ be a set of q cubes in which every face in each one of them has been coloured using one colour in a set $K = \{k_1, k_2, \ldots, k_q\}$ of q colours. The problem we are dealing with is to raise, if it is possible, a pile with the q cubes in such a way that every one of the q colours will appear once in every one of the four sides of the pile. This problem was introduced and solved by De Carteblanche in [1] for the case $q = 4$. We aim to follow the same analysis of the problem in the case $q \geqslant 4$, and pointing to the difficulties in extending the particular solution given when $q = 4$ we introduce an efficient solution for $q \geqslant 4$. Let f_i, b_i, l_i and r_i be, respectively, the colour of cube c_i $(1 \leqslant i \leqslant q)$ appearing in the front, back, left and right side of the pile of cubes. When solving the coloured cubes problem (CCP) it is much useful to establish the *Independence Property* between adjacent sides in the pile. More specifically, every CCP may be solved in two independent steps, first by

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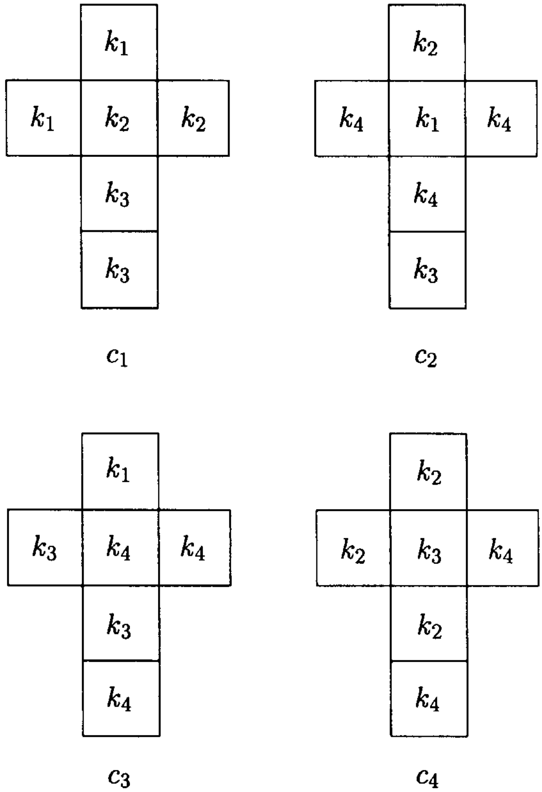

Fig. 1. Set C.

considering a solution corresponding to a pair of opposite sides in the pile (say left and right), and after that by constructing the solution for the other pair of sides (front and back) by a proper rotation of every cube (around an horizontal axis in this case). In the following development we will make use of this property.

Given a CCP, let $G = (V,E)$ be the multigraph with loops allowed associated to it with vertices set $V = K$ and edges set $E = \{(k_i,k_j)\}$ k_i and k_j are the colours assigned to opposite faces in a cube in C. Furthermore, every edge has an integer label in $\{1,2,\ldots,q\}$ which identifies which cube is connected to the edge. In this way $|E| = 3q$ and $|V| = q$. The CCP with $q = 4$ represented in Fig. 1 produces the multigraph G represented in Fig. 2.

A *2-factor* in a graph (multigraph) G is a spanning subgraph of G which produces degree equal to two in every vertex. A 2-factor is *canonical* if it contains every edge label once. By taking graph G in Fig. 2 we can build a pair of edge-disjoint canonical 2-factors; one pair like that is represented in Fig. 3. From now on, every pair of canonical 2-factors considered must be edge-disjoint.

As we will see, any pair of canonical 2-factors will produce a solution for the CCP. In fact, any clockwise traverse of the circuits of a canonical 2-factor produces an

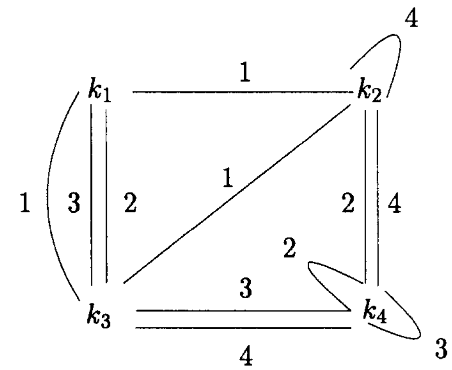

Fig. 2. Multigraph G.

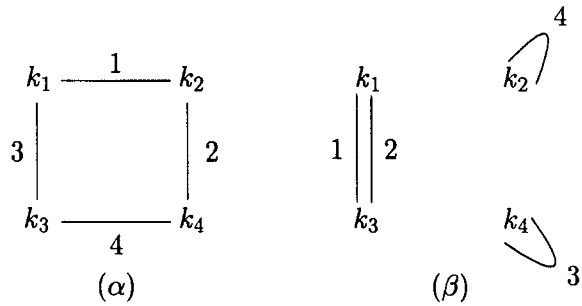

Fig. 3. Two canonical 2–factors in G.

arranging for the cubes in the pile in relation to two opposite faces. So, by taking the 2-factor α in Fig. 3 an arranging is as follows: $f_1 = k_1$, $b_1 = k_2$, $f_2 = k_2$, $b_2 = k_4$, $f_4 = k_4$, $b_4 = k_3$, $f_3 = k_3$ and $b_3 = k_1$. Referring to the 2-factor β an arranging is: $l_1 = k_3$, $r_1 = k_1$, $l_2 = k_1$, $r_2 = k_3$, $l_4 = k_2$, $r_4 = k_2$, $l_3 = k_4$ and $r_3 = k_4$. The conclusion is that taking into account the Independence Property any pair of canonical 2-factors produces a solution for the CCP.

2. A solving method

Difficulties in a CCP turn up when looking for canonical 2-factors because there are 3^q sets of edges candidates to form a canonical 2-factor. While the CCP with four cubes is easily solved by an inspection of its associated multigraph, the general case becomes far from trivial. Our proposal henceforth is to define a type of linear programming problem on the multigraph associated to every CCP which at the end of two successive resolutions produces a pair of canonical 2-factors if they exist. When the first execution is over the set of $2q$ edges which belong to the pair of canonical

2-factors is produced. So, the second execution is devoted to distinguish them; this will define the position of the faces of every cube in relation to the sides of the pile.

Next, we present the formulation of the linear programming problem P. Given the multigraph G associated to a CCP, let nonnegative variables $x_{ij}^t = x_{ji}^t$ denote the edge $(k_i,k_j) = (k_j,k_i)$ in G labelled t. Also, when $i = j$ (a loop in G) two nonnegative variables are defined x_{ia}^t and x_{ib}^t which will take the same value in the solution. Now, the set of constraints is formed by four blocks. The first one imposes that every color appears four times, once in every face of the pile, while the second one imposes that exactly two edges in G are chosen for every cube. Each of the two blocks of constraints consists of q equations. The $3q$ equations in the third block impose the equality between every pair of equivalent variables representing the same edge. Finally, the fourth block is formed by $3q$ constraints bounding the values of all the $6q$ variables. The general form of these four blocks is as follows:

$$\sum_{\substack{\forall j \neq i \\ \forall t}} x_{ij}^t + \sum_{\forall t} x_{ia}^t + x_{ib}^t = 4 \qquad (i = 1,2,\ldots,q),$$

$$\sum_{\forall i<j} x_{ij}^t + \sum_{\forall i} x_{ia}^t = 2 \qquad (t = 1,2,\ldots,q),$$

$$x_{ia}^t - x_{ib}^t = 0, \quad x_{ij}^t - x_{ji}^t = 0 \qquad (i,j = 1,2,\ldots,q \text{ with } i \neq j),$$

$$x_{ij}^t, x_{ia}^t \leqslant 1 \qquad (1 \leqslant i < j \leqslant q;\ 1 \leqslant t \leqslant q).$$

Finally, the objective function is just defined to ensure a suitable solution for the problem, that is to say an integer 0–1 solution. So we take

$$(\max) z = \sum_{\substack{i,j,t=1 \\ i<j}}^{q} k_{ij}^t\, x_{ij}^t + k_{ii}^t\, x_{ia}^t ,$$

k_{ij}^t being any nonnegative constant verifying $k_{ij}^t \neq k_{xy}^w$ whenever $i \neq x$ or $j \neq y$ or $t \neq w$. Clearly, the particular value of the objective function z is not relevant to the problem.

In the final solution, the program will assign $x_{ij}^t = x_{ji}^t = 1$ if the corresponding edge is chosen and $x_{ij}^t = x_{ji}^t = 0$ otherwise. In this way, whenever an admissible pile of cubes exists, the solving of P will produce a set of $4q$ variables (two sets of $2q$ equivalent variables) with a value equal to 1. This set of variables points to the $2q$ edges involved in a pair of canonical 2-factors. The matter now is how to identify efficiently what edges are in the 2-factor corresponding to the left–right sides of the pile (the remaining edges will belong to the front–back sides of the pile). But this identification can be easily done by solving the linear programming P' obtained from the multigraph G' generated by the solution of P. Linear program P' has the same structure defined in P so G' has the same vertices set defined in G and its edges set corresponds to the nonzero variables obtained when solving P. The general formulation of P' is as follows:

$$\sum_{\substack{\forall j \neq i \\ \forall t}} x_{ij}^t + \sum_{\forall t} x_{ia}^t + x_{ib}^t = 2 \qquad (i = 1,2,\ldots,q),$$

$$\sum_{\forall i<j} x^t_{ij} + \sum_{\forall i} x^t_{ia} = 1 \qquad (t = 1,2,\dots,q),$$

$$x^t_{ia} - x^t_{ib} = 0, \qquad x^t_{ij} - x^t_{ji} = 0 \qquad (i,j = 1,2,\dots,q \text{ with } i \neq j),$$

$$x^t_{ij}, x^t_{ia} \leqslant 1 \qquad (1 \leqslant i < j \leqslant q;\ 1 \leqslant t \leqslant q),$$

$$(\max) z = \sum_{\substack{i,j,t \quad 1 \\ i<j}}^{q} k^t_{ij}\, x^t_{ij} + k^t_{ii}\, x^t_{ia}\,.$$

In this way, the set of nonzero variables in a solution of P' generates one of the canonical 2-factors, the remaining zero variables generate the other one.

3. Example

Considering the multigraph G in Fig. 3 its linear program P is as follows:

$$x^1_{12} + x^1_{13} + x^2_{13} + x^3_{13} = 4,$$

$$x^1_{21} + x^1_{23} + x^2_{24} + x^4_{24} + x^4_{2a} + x^4_{2b} = 4,$$

$$x^1_{31} + x^2_{31} + x^3_{31} + x^1_{32} + x^3_{34} + x^4_{34} = 4,$$

$$x^2_{42} + x^4_{42} + x^3_{43} + x^4_{43} + x^2_{4a} + x^2_{4b} + x^3_{4a} + x^3_{4b} = 4,$$

$$x^1_{12} + x^1_{23} + x^1_{13} = 2,$$

$$x^2_{13} + x^2_{24} + x^2_{4a} = 2,$$

$$x^3_{13} + x^3_{34} + x^3_{4a} = 2,$$

$$x^4_{24} + x^4_{34} + x^4_{2a} = 2,$$

$$x^4_{2a} - x^4_{2b} = 0,\ x^2_{4a} - x^2_{4b} = 0,\ x^3_{4a} - x^3_{4b} = 0,$$

$$x^1_{12} - x^1_{21} = 0,\ x^1_{13} - x^1_{31} = 0,\ \cdots\ x^4_{34} - x^4_{43} = 0,$$

$$x^t_{ij}, x^t_{ia} \leqslant 1 \qquad (1 \leqslant i < j \leqslant 4) \qquad (1 \leqslant t \leqslant 4)$$

with

$$(\max) z = x^1_{12} + 2x^1_1 + 3x^2_{13} + \cdots + 12x^3_{4a}\,.$$

Once P has been solved a set of 16 nonzero variables is produced: $\{x^1_{12}, x^1_{21}, x^1_{13}, x^1_{31}, x^2_{13}, x^2_{31}, x^2_{24}, x^2_{42}, x^3_{13}, x^3_{31}, x^3_{4a}, x^3_{4b}, x^4_{34}, x^4_{43}, x^4_{2a}, x^4_{2b}\}$. The multigraph G' associated to this solution appears in Fig. 4. Now, the linear problem P' from G' is given by

$$x^1_{12} + x^1_{13} + x^2_{13} + x^3_{13} = 2,$$

$$x^1_{21} + x^2_{24} + x^4_{2a} + x^4_{2b} = 2,$$

$$x^1_{31} + x^2_{31} + x^3_{31} + x^4_{34} = 2,$$

$$x^2_{42} + x^4_{43} + x^3_{4a} + x^3_{4b} = 2,$$

$$x^1_{12} + x^1_{13} = 1,$$

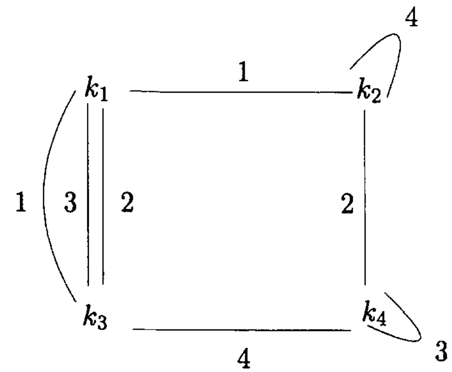

Fig. 4. Multigraph G'

$$x_{13}^2 + x_{24}^2 = 1\,,$$
$$x_{13}^3 + x_{4a}^3 = 1\,,$$
$$x_{34}^4 + x_{2a}^4 = 1\,,$$
$$x_{2a}^4 - x_{2b}^4 = 0,\ \ x_{4a}^3 - x_{4b}^3 = 0\,,$$
$$x_{12}^1 - x_{21}^1 = 0,\ \ x_{13}^1 - x_{31}^1 = 0,\ \cdots\ x_{34}^4 - x_{43}^4 = 0\,,$$
$$x_{ij}^t, x_{ia}^t \leqslant 1 \qquad (1 \leqslant i < j \leqslant 4) \qquad (1 \leqslant t \leqslant 4)\,,$$
$$(\max) z = x_{12}^1 + 2x_{13}^1 + 3x_{13}^2 + \cdots + 8x_{4a}^3\,.$$

Once P' has been solved the set $\{x_{13}^1, x_{13}^2, x_{2a}^4, x_{2b}^4, x_{31}^1, x_{31}^2, x_{4a}^3, x_{4b}^3\}$ of eight nonzero variables is obtained. So, this set contains the first 2-factor; the other one is given by the remaining zero variables.

Reference

[1] F. de Carteblanche, The coloured cubes problem, Eureka (1947) 9–11.